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Toda Theories, Matrix Models, Topological Strings, and

$N = 2$ Gauge Systems

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We consider the topological string partition function, including the Nekrasov deformation, for type IIB geometries with an A_{n-1} singularity over a Riemann surface. These models realize the $\mathcal{N} = 2$ $SU(n)$ superconformal gauge systems recently studied by Gaiotto and collaborators. Employing large N dualities we show why the partition function of topological strings in these backgrounds is captured by the chiral blocks of A_{n-1} Toda systems and derive the dictionary recently proposed by Alday, Gaiotto and Tachikawa. For the case of genus zero Riemann surfaces, we show how these systems can also be realized by Penner-like matrix models with logarithmic potentials. The Seiberg-Witten curve can be understood as the spectral curve of these matrix models which arises holographically at large N. In this context the Nekrasov deformation maps to the β-ensemble of generalized matrix models, that in turn maps to the Toda system with general background charge. We also point out the notion of a double holography for this system, when both n and N are large.

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1. Introduction

The geometry of $\mathcal{N} = 2$ supersymmetric gauge theories in 4 dimensions has played a prominent role in understanding strong coupling aspects of gauge theories. In particular the geometry underlying the vacuum solution of $SU(2)$ gauge theory with matter discovered by Seiberg and Witten [1] and its various subsequent generalizations have been a very fruitful arena. The embedding of these gauge theories in string theory has led to valuable insights into both gauge theory dynamics as well as into string dualities. Typically one finds a construction in string theory that realizes the gauge system and then uses string dualities to find the vacuum solution.

There are two prominent ways this has been done. The first is in the form of geometric engineering of gauge theories [2]. Here one considers a local ADE geometry of a Calabi-Yau 3-fold in type IIA to engineer the gauge theory and uses the mirror symmetry to find the solution of it in terms of the mirror Calabi-Yau of type IIB. For some special gauge groups and quivers the mirror CY data is captured by a Riemann surface which gets identified with the Seiberg-Witten curve. Moreover, by another duality, this gets mapped to NS 5-branes of type IIA wrapping the curve and filling the spacetime [2]. The topological B-model on the mirror Calabi-Yau not only yields the vacuum geometry and thus the $\mathcal{N} = 2$ prepotential (given by genus zero partition function of the topological string), but also computes induced gravitational corrections (coming from genus $g > 0$ topological string amplitudes) of the form [3,4]

$$\int d^4\theta \, \mathcal{F}_g(\mu) \mathcal{W}^{2g}.$$

Here $\mathcal{W}$ is the self-dual graviphoton chiral field, and $\mathcal{F}_g(\mu)$ denotes the genus g amplitude of topological strings, written as a function of the complex structure moduli μ (in the B-model). All these quantities are elegantly combined in terms of the topological string partition function

$$Z(\mu; g_s) = \exp \sum_{g \geq 0} g_s^{2g-2} \mathcal{F}_g(\mu). \tag{1.1}$$

It was shown in [5] that these corrections are also directly computable in the gauge theory setup using equivariant instanton calculus of a $U(1) \times U(1)$ subgroup of the rotation group $SO(4)$ acting on Euclidean space-time. Moreover these gravitational corrections were generalized to include a one-parameter deformation given by the anti-self dual twisted graviphoton field. This refined partition function $Z(\mu; \epsilon_1, \epsilon_2)$ of Nekrasov reduces to the topological string (1.1) in the case $\epsilon_2 = -\epsilon_1 = g_s$.

In another approach [6], one realizes the gauge system in terms of D4 branes suspended between NS 5-branes, and one obtains the strong coupling solution by lifting the geometry to M-theory, where the relevant geometry emerges in the form of the M5 brane wrapping the curve. In fact, a large class of such gauge theories that are conformal, together with their mass deformations, were constructed and studied in [6] by considering a system of N M5 branes wrapping a sphere or a torus.

Recently these constructions were generalized by Gaiotto [7] to the case of arbitrary Riemann surfaces, and were also studied holographically in [8]. Furthermore a surprising conjecture was made by by Alday, Gaiotto and Tachikawa in [9], that for the case of two M5 branes, *i.e* a $SU(2)$ gauge theory, the Nekrasov partition functions are simply given by the chiral blocks of a Liouville theory on the corresponding curve. More precisely the AGT conjecture says that:

(1) the mass deformations are given by Liouville vertex operators carrying the corresponding momentum;

(2) the Liouville momenta in the intermediate chanels characterize the Coulomb branches of the gauge theory;

(3) the position of the vertex operators correspond to gauge coupling constants; and

(4) the choice of the background charge of the Liouville CFT is related to Nekrasov's deformations ϵ_1, ϵ_2 of the topological string partition function.

By now there is strong evidence [10,11] that this correspondence with Liouville for the $SU(2)$ case, and its generalization to Toda for the $SU(n)$ case is valid. The main question is what is the explanation of such a duality. In this paper we offer a simple stringy explanation based on large N dualities of topological strings.

The basic ingredients of our derivation is as follows: We realize the gauge theories of interest by geometrically engineering them in type II theories and ask how one can compute topological string amplitudes for such backgrounds. It has been known that for toric geometries a powerful approach for such computations is through the study of large N dualities for topological strings [12], where topological string amplitudes get related to gauge theories on branes. In particular in the type IIA setup this relates topological A-model string amplitudes to Chern-Simons amplitudes. For example, the partition function of A-model on the resolved conifold, gets related to Chern-Simons theory on S^3. The B-model analog of these large N dualities where proposed in [13], motivated from their embedding in superstrings [14]. It was shown how the matrix models on the gauge theory

side, which is the effective theory on the B-branes, compute amplitudes of the closed string side (where the closed string geometry was captured by the resolvent of the large N limit of the matrix model). Moreover the analog of brane probes correspond to free fermions living on the spectral curve [15].

The study of D-brane probes on both sides of the duality leads to a local relation between open string computations, such as knot invariants, with string amplitudes in the presence of brane probes. The topological vertex formalism [16] is a theory which captures the correlations of such D-branes, which can be computed using this duality. The closed string amplitudes are the special case of such amplitudes where no brane is inserted, *i.e.* its vacuum amplitude.

To use this machinery we are naturally led to ask which CY geometries realize the $\mathcal{N} = 2$ CFT and their deformations that are relevant for the AGT conjecture. It turns out that the simplest examples of such $\mathcal{N} = 2$ CFT's were already constructed in the type IIA setup in [17] as a precursor to the more general construction embodied in the topological vertex. This includes for example the case of $SU(2)$ with 4 hypermultiplet doublets. The reason these constructions were simpler was that by a series of transitions one could get rid of all the 2-cycles in the type IIA geometry and describe the entire geometry using transitions for geometries with 3-cycles only, with branes wrapped on them.

However the expansion point suitable for [17] corresponds to the large Coulomb branch parameter for the $SU(2)$, whereas the expansion relevant for the AGT conjecture involves expansion near zero Coulomb parameters. Moreover, the answer in [17] is directly relevant for the 5d theories compactified on a cricle. To obtain the purely 4d answer we need to take the limit of small radius of the circle. This leads us to the study of the B-model topological string for the relevant geometry, which is more suitable for the 4d limit. It is thus natural to look for a simpler version of [17] relevant for the AGT conjecture and that is what we focus on in this paper.

There are two alternative derivations we find. In the first approach we utilize the well-known connections between matrix model and topological strings on Calabi-Yau [13] on the one hand, and the relation between matrix model and Toda systems [18] on the other hand, to derive the AGT conjecture. Namely, we use the matrix model directly to engineer the corresponding Calabi-Yau. Here the CY of interest is obtained by geometric transitions from the geometry on which the branes live, induced by having a large number of branes (corresponding to the large N limit of matrix model). This derivation is simplest

for an A_{n-1} singularity over a sphere with punctures, although it can, with some subtleties, be generalized to arbitrary curves.

In this derivation, the starting geometry before transition corresponds to a subspace of the masses and Coulomb branch parameters where the Toda momenta are additively conserved. This Calabi-Yau geometry is fixed in terms of the data of this restricted masses and Coulomb parameters. In the case of $SU(2)$ this geometry involves a number of conifold points. Blowing the conifold points and placing topological B-branes in this background give rise to a matrix model, whose action can be read off from the CY geometry. The open string action involves a sum of Penner-like potentials. The momentum non-conserving part of the Liouville amplitudes, which lead to more general mass and Coulomb moduli, arise by going to the large N limit of the matrix model and distributing the matrix eigenvalues at the different critical points (corresponding to deformations of the conifolds). The correlation functions of the corresponding Toda theory get mapped to A_{n-1} quiver matrix model. The well known direct map between matrix models and Liouville theories [18] completes the derivation of AGT.

Our second derivation, namely the brane probe approach, is more conceptual. We work with the brane probes of CY geometries and find a nice way to study their effective action near the A_{n-1} singularity, where the Coulomb parameter is near zero. For such geometries instead of one type of D-brane probe we get n of them. Moreover, the brane probes, which are well known to relate to fermions [15], now become a system of n free fermions. Bosonization of these fermions leads to the n Toda fields of the $U(n)$ theory. It is important that we have an extra $U(1)$ in the theory which resolves some of the puzzles in AGT prescription associated with a missing $U(1)$. However, it turns out that the fact that these n fermions live on the A_{n-1} singularity makes them interacting in an interesting, non-commutative way. This involves deforming the theories by insertion of operators which correponds to currents of the $U(n)$ theory along cycles. This interaction gets mapped, upon bosonization, to the Toda potential. Moreover introduction of mass parameters corresponds to a geometry, which before transition simply multiplies the fermionic wave function by a pole term, with residue given by mass. We show how this arises and explain why it maps, upon bosonization, to the vertex operator for the Toda field with momentum in the fundamental weight of the corrsponding gauge group.

Furthermore we explain that the Nekrasov deformation of topological string amplitude correspond to the spectral flow operator for the fermionic sea. In the matrix model approach, the Nekrasov deformation gets mapped to considering a 'β-ensemble' [19] (for

some recent literature see [20]) where $\beta = -\epsilon_2/\epsilon_1$ and the Vandermonde determinant is raised to the powers of 2β instead of 2.

The organization of this paper is as follows: In section 2 we recall the relation between matrix models and topological strings on the one hand, and matrix models and Toda theories on the other. Using this we show how the AGT conjecture arises for genus 0, and can be extended to higher genera. In section 3, we start the alternative brane probe derivation by recalling the relevant Calabi-Yau geometries, both in the type IIA set up and their mirror in type IIB. In addition we recall some basic facts about brane probes for topological strings and their relation to fermions. In section 4, we discuss branes for the A_{n-1} singularities and how bosonization of the system relates it to Toda. Furthermore we describe geometries which 'chop-off' the A_{n-1} singularities to pieces and their effect on D-brane probes. This leads us to the alternative derivation of AGT conjecture. In section 5 we describe the Nekrasov deformation, both in the brane picture (which leads to background charge for Toda theory) and in the matrix model approach. Finally in section 6 we end with some issues for further study.

2. Matrix Model and the Emergence of Geometry

Matrix models are known to provide powerful descriptions of topological string B-models on special classes of Calabi-Yau geometries. We will see that this technique can also be used in the case of the geometries that are relevant for $\mathcal{N} = 2$ superconformal gauge theories. Matrix models thus provide a direct bridge between the Calabi-Yau geometries of interest, and Liouville and Toda systems.

This connection to matrix models is not surprising. On the one hand we already know, through geometric transitions, that matrix models do describe local topological B-models [13]. On the other hand it is also known that the collective field for matrix models is related to Toda systems [18]. The connection between the two links the rank N of the matrices with the number of insertion of screening operators on the Toda side. The new ingredient will be that the correlation functions of the Toda system are naturally considered in the large N limit in order to connect to the gauge theory quantities.

We begin the discussion below with the case of the single matrix model and recall its connection with local Calabi-Yau geometries, and at the same time connect it to the Liouville description. We then discuss its extension to multi matrix models. Note that compared to the applications considered in [13], where the number of matrix fields was

related to the number of gauge factors in the quiver, in the current applications the number of matrices gets related to the rank of the gauge group. In geometric engineering terminology this is the same as exchange of the base and the fiber geometry.

2.1. Matrix models and CFT

For convenience let us concentrate first on the $SU(2)$ case that corresponds to a single $N \times N$ matrix. The generalization to $SU(n)$ is rather straightforward. In general the partition function of such a matrix model takes the form

$$Z = \int_{N \times N} d\Phi \ \exp \frac{1}{g_s} \mathrm{Tr}\, W(\Phi). \tag{2.1}$$

Here and subsequently, all variables are complex and the integral is a suitable contour integral in the space of complex matrices, *e.g.* the real subspace of hermitean matrices. The matrix model describes topological B-model string theory on a Calabi-Yau geometry of the form

$$uv + F(x, z) = 0,$$

where the equation $F(x, z) = 0$ describes some algebraic curve Σ in $\mathbf{C}^2$. In the matrix model Σ is identified with the spectral curve and is given by a double cover of the z-plane of the form

$$F(x, z) = x^2 - W'(z)^2 + f(z) = 0. \tag{2.2}$$

Saddle points of the matrix integral correspond to a distribution $N = N_1 + N_2 + \dots$ over the critical points $y_1, y_2, \dots$ of the potential $W(z)$. Such a saddle point determines the so-called quantum correction $f(z)$ in (2.2).

This connection between matrix models and CY geometries was explained in [13]. The idea was based on geometric transition where one considers the geometry before transition given by

$$uv + x^2 - W'(z)^2 = 0,$$

and the conifold singularities at $u = v = x = W'(z) = 0$ are resolved into $\mathbf{P}^1$'s. We then distribute N copies of B-branes over these $\mathbf{P}^1$'s and this determines the geometry after transition.

The relation between matrix model and the topological B-model on (2.2) is simply the large N duality in the context of topological string. The open B-model is given by a matrix model with potential $W(\Phi)$ and thus the partition function of the matrix model computes

the closed B-model topological string amplitudes on the emergent geometry given by (2.2). In other words, the matrix model computes the amplitudes of the closed topological string on a CY which has undergone a geometric transition.

By tuning the potential and the quantum correction, a general hyperelliptic curve $x^2 = P(z)$ can be engineered. The higher genus corrections can be captured in terms of collective string field theory $\phi(z)$ living on the spectral cover. This scalar field gives a natural connection to two-dimensional chiral conformal field theory. In fact, it is known that this collective field for matrix models is closely related to Liouville theory [18]. Let us briefly review this connection.

The matrix model partition function can be written in terms of the eigenvalues z_I of the matrix Φ as

$$Z = \int d^N z \prod_{I<J} (z_I - z_J)^2 \exp \sum_I \frac{1}{g_s} W(z_I)$$

This expression can be recognized as a correlation function of a free conformal scalar field theory [18]. In particular, the Vandermonde determinant of the eigenvalues has an interpretation in terms of an (integrated) N-point function of vertex operators $e^{i\phi}$ at positions z_I, and the potential term looks like a background gauge field coupling to the current $\partial\phi$. So, we can rewrite the matrix integral as

$$Z = \langle N| \int d^N z \, e^{i\phi(z_1)} \cdots e^{i\phi(z_N)} \exp\left(\oint_\infty dz \, \frac{1}{g_s} W(z) \partial\phi(z) \right) |0\rangle \qquad (2.3)$$

Here $|N\rangle$ denotes the vacuum with total momentum/charge N,

$$\oint \partial\phi |N\rangle = N|N\rangle.$$

Correlation functions like (2.3) with a total net charge N we will denotes as $\langle \cdots \rangle_N$. Using this notation the partition can be written in exponentiated form as the generating function

$$Z = \left\langle \exp\left(\int_{-\infty}^{\infty} dz \, e^{i\phi(z)} \right) \exp\left(\oint_\infty dz \, \frac{1}{g_s} W(z) \partial\phi(z) \right) \right\rangle_N$$

The scalar $\phi(z)$ appears directly in the large N limit of the matrix model as the collective field of the eigenvalues. Matching the normalizations, it is given by[1]

$$\phi(z) = \frac{1}{g_s} W(z) + 2 \operatorname{Tr} \log(z - \Phi).$$

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[1] Here normalizations differ. The Liouville field has a kinetic term with coupling one; the collective field of matrix models has a coupling $1/g_s^2$ appropriate for closed string field. So, in general: $\phi_{Liouville} = g_s \phi_{string}$. We sometimes switch between these conventions.

or, equivalently,

$$\partial\phi(z) = \frac{1}{g_s}W'(z) + 2\,\mathrm{Tr}\frac{1}{z - \Phi}.$$

At this point it is perhaps also good to recall for later use that the loop equations of the matrix model, which give rise to the spectral curve (2.2), are a reflection of the invariance of the matrix integral (2.1) under infinitesimal diffeomorphisms $\delta\Phi = \Phi^{n+1}$. These transformations are generated by the modes L_n of the stress-tensor $T(z) = \frac{1}{2}(\partial\phi)^2$. The spectral curve can thus be written as

$$x^2 = g_s^2\langle\partial\phi(z)^2\rangle.$$

It is convenient to rewrite this CFT representation using free fermions. Hereto we consider two chiral Dirac fermions ψ_1, ψ_2 and their conjugates ψ_1^*, ψ_2^*. These can be bosonized in terms of two scalar fields as

$$\partial\phi_1 = \psi_1^*\psi_1, \qquad \partial\phi_2 = \psi_2^*\psi_2.$$

In fact, it will be convenient to introduce the odd and even combinations under the $\mathbf{Z}_2$ exchange $\psi_1 \leftrightarrow \psi_2$

$$\phi_\pm = \phi_1 \pm \phi_2.$$

The fermions ψ_i, ψ_i^* form a level one representation of the $U(2)$ affine current algebra. The $U(2) = U(1) \times SU(2)$ currents acting on the two fermions can also be bosonized. The $U(1)$ current is given by $\partial\phi_+$ and essentially decouples from the problem, whereas the triplet of $SU(2)$ currents is expressed in terms of the field ϕ_- as

$$J_+ = e^{i\phi_-}, \qquad J_3 = \partial\phi_-, \qquad J_- = e^{-i\phi_-}.$$

We will now concentrate on the $SU(2)$ sector and the field ϕ_-, often dropping the $-$ subscript.

In terms of these $SU(2)$ fermion currents the matrix model takes the form

$$Z = \left\langle \exp\left(\int dz\, J_+(z)\right) \exp\left(\oint_\infty dz\, W(z)J_3(z)/g_s\right) \right\rangle_N$$

Notice that the two contours along which we integrate the currents are different. We integrate J_+ along the real axis, whereas J_3 is integrated around ∞.

Now, general one-matrix model leads to general hyperelliptic curves like (2.2), but here we are actually interested in the simple A_1 geometry

$$uv + x^2 = 0,$$

with z a free parameter. So the Calabi-Yau geometry is $A_1 \times \mathbf{C}$, with z being the local parameter on the line $\mathbf{C}$. The corresponding curve is in that case simply given by

$$F(x, z) = x^2 = 0,$$

or its deformation

$$x^2 - p^2 = 0.$$

From the point of view of the matrix model this A_1 geometry can be obtained as a limiting case (double scaling limit) of (2.2) with vanishing matrix potential $W = 0$. To this end one introduces a small quadratic potential $W = \epsilon z^2$, fills the critical point with N eigenvalues and then takes the simultaneous limit $N \to \infty$ and $\epsilon \to 0$. This produces an (almost) constant eigenvalue density. This limit can be thought of as describing the local situation at a point of a general spectral curve. This construction is well-known in mathematical physics to capture the so-called universal behaviour of random matrix models [21,22]. Pictorially this can be thought of by taking the Wigner circle and deforming it to a very long and narrow ellips, ending up with a double line.

In fact, we can be a bit more general and choose the limiting case $W = pz$. This choice of potential corresponds to a background value for the scalar field given by $\phi(z) = pz$, or $\partial \phi = p$. So, there is constant background momentum p flowing through the z line. In this case there is no quantum correction, and the effective curve is given by

$$x^2 - p^2 = (x + p)(x - p) = 0.$$

This is the deformed A_1 singularity. In the limit $p \to 0$ we recover the original singularity.

Coming back to the matrix model and its bosonic or fermionic CFT reincarnations, we learn that the special case of vanishing W corresponds to the geometry $A_1 \times \mathbf{C}$. Note that in terms of the bosonic conformal field theory this special case corresponds to a CFT action

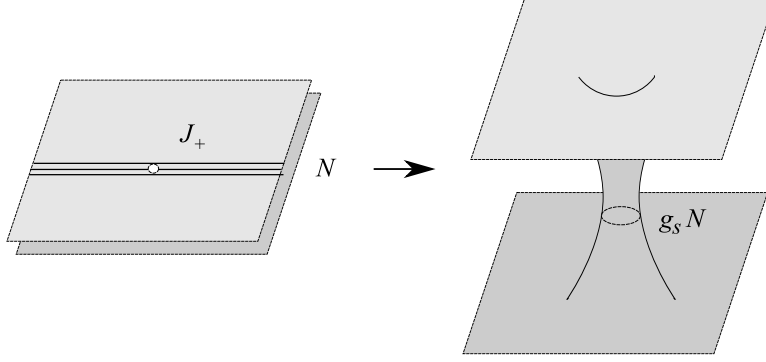$$S = \int d^2 z \, \partial \phi \overline{\partial} \phi + \int dz \, e^{i\phi(z)}.$$

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Fig. 1: Insertions of N copies of the screening charge $\int J_+$ result in the $N \to \infty$ limit in opening up the double points of the curve to a size $g_s N$.

We want to think of this as a chiral version of Liouville theory at $c = 1$ (there is no backgrund charge, yet), where the usual surface integral of the Liouville potential has been replaced by a line integral of the screening charge. We think of evaluating this as a Coulomb gas model, perturbing in the screening charge $\int J_+$, where the rank N of the matrix model corresponds to the number of insertions of the screening charge. Note that this sheds a different light on the spectral curve. The resolution of double points can now be thought of in terms of condensation of the screening charge integrals, as illustrated in fig. 1. In order to make sense of this picture it helps to consider correlation functions.

2.2. D-brane insertions and vertex operators

It is easy to incorporate D-brane insertions in the CFT or matrix model. As we will review in section 3, in the fermionic model placing m B-branes corresponds to the insertion of the operator

$$V_m = e^{im\phi_1} = e^{im\phi_+/2} \cdot e^{im\phi_-/2},$$

which up to the $U(1)$ factor is equivalent to the insertion of

$$V_m = e^{im\phi_-/2}.$$

Here we use a normalization of the kinetic term of the fields $\phi_\pm$ so that the vertex operator V_m has conformal dimension $m^2/4$. In particular the screening operator is given by $e^{i\phi_-}$ and has dimension 1.

With this normalization the vertex operator V_m is given in the matrix model by

$$V_m(q) = \det(\Phi - q)^m.$$

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So, we find that a general correlator of D-branes in the topological B-model of the $A_1 \times \mathbf{C}$ geometry gives rise to the matrix integral of the form

$$\langle V_{m_1} \cdots V_{m_k} \rangle_N = \int_{N \times N} d\Phi \prod_{i=1}^{k} \det (\Phi - q_i)^{m_i} .$$

This expression can be written in terms of the eigenvalues z_I of the matrix Φ as

$$\langle V_{m_1} \cdots V_{m_k} \rangle_N = \int d^N z \prod_{I<J} (z_I - z_J)^2 \prod_{i,I} (z_I - q_i)^{m_i}$$

Here we recognize the Coulomb gas formulation. Note that there should be a charge m_0 at infinity given by

$$m_0 = -\sum_{i=1}^{k} m_i - N.$$

So, we are actually dealing with a $k+1$ point function on $\mathbf{P}^1$. The action of $SL(2, \mathbf{C})$ on the sphere can be used to put these insertions in general position.

2.3. The spectral curve

Now the question is whether we can use the standard matrix model techniques to solve these expressions. This turns out indeed to be the case.

The insertions of the determinants $V_m(q)$ can of course be reformulated as a logarithmic action

$$W(\Phi) = \sum_{i=1}^{N} g_s m_i \log (\Phi - q_i) .$$

Written in this way it is clear we are dealing with a 'multi-Penner' matrix model, with an action given by k logarithms.

We now want to study the g_s expansion of these expressions. We can treat this model as any other random matrix model and study its large N limit, keeping the 't Hooft parameter $g_s N$ finite. We should also send the matrix model/Liouville parameters $m_i \to \infty$ keeping the *gauge theory* masses

$$\mu_i = g_s m_i$$

finite.

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General considerations will tell us that in this limit the genus zero contribution will be captured by an effective algebraic spectral curve Σ. In the present $SU(2)$ case it will be given by a double cover of the sphere. Let us now describe this curve in more detail.

The curve Σ is determined by the distribution of the N eigenvalues over the critical point $W'(y_i) = 0$. In this case we have

$$W'(z) = \sum_{i=1}^{k} \frac{\mu_i}{z - q_i}. \tag{2.4}$$

So there $k-1$ critical points and the saddle-point is determined by the filling fractions

$$\nu_i = g_s N_i, \qquad i = 1, \ldots, k-1.$$

Of course, the total sum of the ν_i is fixed to be

$$\sum_{i=1}^{k-1} \nu_i = g_s N = -\sum_{j=0}^{k} \mu_j.$$

It is not difficult to describe the corrsponding spectral curve. First of all, as in any matrix model, the quantum correction is given by the saddle point value of the expression

$$f(z) = \left\langle g_s \sum_{I} \frac{W'(z_I) - W'(z)}{z_I - z} \right\rangle_N.$$

(Here z_I are the eigenvalues.) Since $W'(z)$ is a sum of simple poles at $z = q_i$, one easily verifies that also $f(z)$ takes that form

$$f(z) = \sum_{i=1}^{k} \frac{c_i}{z - q_i}. \tag{2.5}$$

The coefficients c_i are determined by the filling fractions ν_i. They satisfy one constraint, that follows directly from the relation $\sum_I W'(z_I) = 0$,

$$\sum c_i = 0.$$

This leaves indeed $k-1$ free combinations of the parameters c_i that can be traded for the filling fractions ν_i.
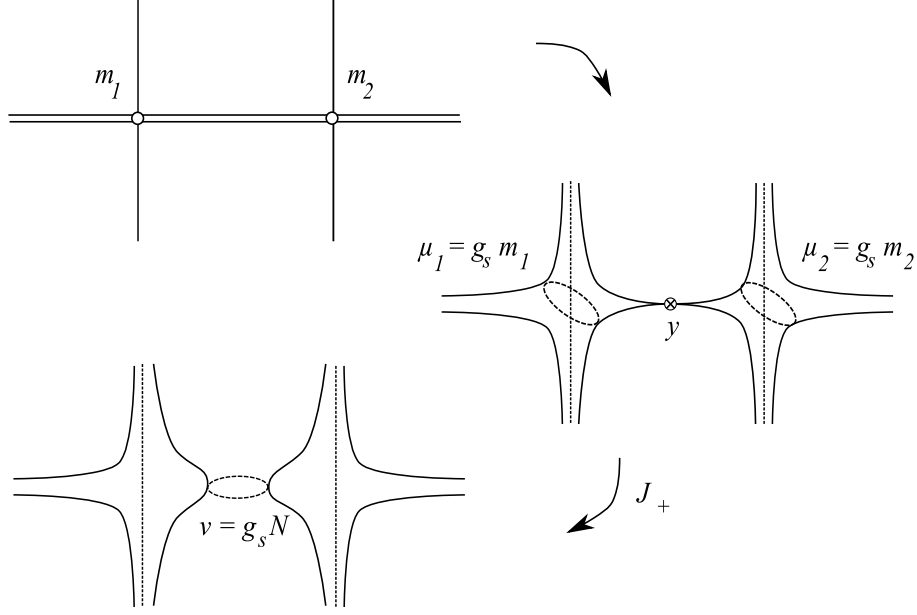
Fig. 2: Condensation of eigenvalues at the double points of the curve $x^2 = W'(z)^2$ with $W'(z) = \sum \mu_i/(z-q_i)$ opens them up into branch points of size $\nu_i = g_s N_i$. In Liouville theory this corresponds to the inclusion of the screening operator $\int J_+$.

These parameters have a nice geometric description as depicted in fig. 2. They open up the double points $z = y_i$ of the curve $x^2 = W'(z)^2$ into branch points, creating a higher genus curve. The parameters ν_i measure the size of the branch cuts

$$\oint_{\mathcal{C}_i} g_S \partial \phi = \oint_{\mathcal{C}_i} x dz = \nu_i, \tag{2.6}$$

where the contours $\mathcal{C}_i$ encircles the ith cut.

Combining (2.4) and (2.5) we find that the spectral curve takes the form

$$x^2 = \frac{P_{2k-2}(z)}{\Delta_k(z)^2}. \tag{2.7}$$

with

$$\Delta_k(z) = \prod_{i=1}^{k}(z - q_i).$$

This gives the right counting of moduli. The numerator $P_{2k-2}(z)$ is a general polynomial of degree $2k-2$. Its $2k-1$ coefficients are the moduli of the curve. They encode two types of data. First of all we have the $k+1$ residues μ_i^2 of the double poles of the quadratic

differential $\phi_2(z)$ at $z = q_i$. Here we should also include the double pole at $z = \infty$ with residue μ_0^2.

Secondly, we have the $k - 1$ independent coefficients c_i, or equivalently the moduli ν_i. Since the sum $\sum \nu_i$ is fixed in term of the masses, this gives $k - 2$ extra moduli

$$a_i = \nu_{i+1} - \nu_i, \qquad i = 1, \ldots, k - 2.$$

These a_i have an interpretation of Coulomb parameters of the corresponding $\mathcal{N} = 2$ gauge theory.

So, let us summarize what we have learned as far as the Liouville theory is concerned. We considered the correlation function of vertex operators $e^{im_i\phi}$ in $c = 1$ Liouville theory. The insertions of the Liouville potential $\int e^{i\phi}$ can be studied in the large N limit, where N is the number of insertions, if we take at the same time the m_i large. These insertions cluster around the critical points of the potential induced by the vertex operators, pinning the eigenvalues. These clusters lead to branch cuts opening up, leaving us with an *effective* CFT living on the spectral cover (2.7).[2]

Because of these branch points the scalar field ϕ is not well-defined on the z-plane. It transforms as $\phi \to -\phi$ if we go through the cut. Only on the double cover (2.7) does it become single valued. In fact, the deck transformation that interchanges the two sheets acts as $\phi \to -\phi$. This is obvious as ϕ started its life as the odd combination $\phi_1 - \phi_2$. The $\mathbf{Z}_2$ involution is simply the action of the Weyl group of $SU(2)$. This is consistent with the fact that Liouville theory does make sense downstairs, because here we are daling with the holomorphic square root or *holomorphic blocks*. These typically have monodromies.

Viewed from this perspective the gauge parameters a_i defined in (2.6) measure the charge flowing through the branch cut or neck connecting the two sheets. The mass parameters m_i in a similar fashion measure the charge flowing through the long thin neck that corrresponds to a puncture.

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[2] By general arguments the scalar field living on the spectral curve is the Kodaira-Spencer field studied by Eynard and collaborators, see *e.g.* [23,24].

2.4. Higher genus curves

A similar story appears for higher genus curves. Again we restrict our attention to the A_1 case.

The description can be stated as follows. The Calabi-Yau geometry is given by a family of A_1 singularities fibered over a Riemann surface Σ with local coordinate z. So, to begin with, we have a geometry of the form

$$uv + x^2 = 0,$$

for arbitrary z on the curve. This is dual to two M5 branes wrapping Σ. We next consider deforming the CY geometry to

$$uv + x^2 - W'(z)^2 = 0.$$

where $W'(z)dz$ is a well defined one-form on Σ. In particular we choose k points on Σ where $W'(z)$ has a pole with residue m_i. This corresponds to the bifundamental masses and also 'partitioning' the curve to gauge factors. Note that the curve is so far factorizable:

$$x^2 - W'(z)^2 = 0.$$

To obtain the most complete type IIB geometry for this gauge system we should also allow arbitrary Coulomb parameters. We split this into two parts: The Coulomb branch parameters which keep the curve factorized, and the ones that do not. The Coulomb branch parameters can be realized by having loop momenta flowing through each of the g A-cycles of the Riemann surface. Let ω_j denote a basis for the g holomorphic 1-forms; ω_j has period 1 around the cycle A_j and 0 around the other cycles A_k. Furthermore, let the momentum flowing through the cycles A_j be p_j. Including both the masses and the momenta through A-cycles we find that

$$W'(z)dz = \sum_{i=1}^{k} m_i \partial_z \log E(z, q_i) + \sum_{j=1}^{g} p_j \omega_j.$$

Here $E(z, q_i)$ is a 'function' on the Riemann surface which vanishes to first order as $z \to q_i$ and is known as the prime form. This is the effective geometry as seen by the free part of the Liouville field. We are now left to deform these further to the most general Coulomb parameters. Instead, we *induce* these deformations by large N transition, as in genus 0

case. Namely we note that our CY has conifold singularities and resolving the singularities and placing branes there gives an effective matrix model as the open string sector of the theory. The choice of distributing the eigenvalues of the matrix at different critical points of $W'(z)$ gives us the extra Coulomb parameters. However now we use the fact that matrix model partition function (the open string side) gives the closed topological string partition function, *i.e.*, the partitions function for CY with the full general Coulomb parameters, which is also dual to 5-branes on the corresponding curve. Finally, the equivalence of this generalized matrix model with Liouville, leads us to the derivation of the AGT conjecture (modulo the Nekrasov deformation which will be covered in section 5).

Note that as in [13] placing branes at the resolved S^2 gives a corresponding potential $W(z)$ as input for a generalized matrix model. These are generalized matrix models in the sense that the eigenvalues of the matrix live on the curve Σ.[3] Note that the Vandermonde determinant gets replaced by the corresponding Green's functions on Σ that compute the N-point functions of J_+ insertions of the Liouville theory. As far as the perturbative expansion is concerned it doesn't matter much how we pick the contours of the J_+ integrations, as longs as they go through all critical points.

This gives the right number of moduli. Note that W' has $k + 2g - 2$ zeroes as it is a one-form on a genus g curve with k poles. Each such zero gives rise to a conifold geometry. However we are free to choose arbitrary insertions of $\int J_+$ over contours that go through critical points of the W'. There are $k + 2g - 3$ independent cycles. (The 'net' number of branes is zero, and thus we have one less modulus than the number of critical points of W.) In this way, as we noted before, we end up making the conifold geometry undergo a transition and we open up such cuts. This gives altogether $k + 2g - 3$ additional parameters, depending on the filling fractions. Together with the original g momenta p_i we end up with a total of $k + 3g - 3$ parameters, which is the expected number of parameters for the Coulomb branches of the gauge system.

2.5. Multi-matrix models and Toda systems

It is not difficult to generalize this picture from $SU(2)$ to $SU(n)$, in which case we are dealing with an A_{n-1} singularity

$$uv + x^n = 0.$$

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[3] In principle such matrix models can be described by going to a cover theory, such as the upper half plane, and modding by suitable subgroups needed to construct Σ. This makes it clear why the eigenvalues, which describe normal deformations of the B-brane, live on the curve.

The corresponding Riemann surface is given by

$$F(x, z) = x^n = 0,$$

and its general deformation is

$$\prod_{a=1}^{n} (x - p_a) = 0.$$

Spectral curves that are n-fold covers naturally emerge from quiver matrix models. In fact, these models can be constructed for any ADE type singularity. Let us briefly summarize this construction. If the Lie algebra has rank r, we have r matrices Φ_a of rank N_a, one for each node of the Dynkin diagram, with bifundamental fields connecting them. Each matrix Φ_a has an individual potential W_a. After integrating out the bifundamentals, the partition function can be written in terms of the corresponding eigenvalues $z_{a,I}$ as (here $a = 1, \ldots, r$ and $I = 1, \ldots, N_a$)

$$Z = \int \prod_{a,I} dz_{a,I} \prod_{(a,I) \neq (b,J)} \left(z_{a,I} - z_{b,J} \right)^{e_a \cdot e_b} \exp \sum_{a,I} \frac{1}{g_s} W_a(z_{a,I}). \tag{2.8}$$

Here e_a are the simple roots and $C_{ab} = e_a \cdot e_b$ is the Cartan matrix.

In the case of A_{n-1} quiver matrix model there are $r = n - 1$ matrices and the CY geometry takes the form [25]

$$uv + \prod_{a=1}^{n} \left(x - t_a(z) \right) + \ldots = 0.$$

Here $t_1 = 0$ and $t_a = \sum_{b=1}^{a-1} W_a'(z)$, and the ellipses denote the quantum resolution of the double points. Again, taking the limiting case of linear potentials gives rise to the geometry of the deformed A_{n-1} singularity

$$uv + \prod_{a=1}^{n} (x - p_a) = 0.$$

This multi-matrix integral can be similarly cast in a CFT notation, where now we have r free chiral scalar fields. It is convenient to use a basis $\phi_a(z)$ that corresponds to the simple roots of the Lie algebra. We then recognize that the integrand in expression (2.8) has the form of a correlation function of a gas of vertex operators $e^{i\phi_a}$ at positions $z_{a,I}$.

$$Z = \left\langle \int \prod_{a,I} dz_{a,I} \prod_{a,I} \exp i\phi_a(z_{I,a}) \right\rangle_{\{N_a\}}. \tag{2.9}$$

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Here N_a denotes the total charge, which is now also an r-dimensional vector. The generating function for these correlators is given by the $c = n - 1$ Toda potential

$$\left\langle \exp \int dz \sum_a e^{i\phi_a(z)} \right\rangle_{\{N_a\}}.$$

Of course, for the A_{n-1} case there is a similar fermionic description in terms of n Dirac fermions ψ_k, ψ_k^*. This gives a realization of level one $U(n)$ Kac-Moody algebra. In the geometric set-up these fermions can be seen to live on the n sheets described by the deformed A_{n-1} singularity, as we will discuss in detail in the next section.

In a similar fashion as for the $SU(2)$ case vertex operator insertions can be included. Chopping off the A_{n-1} singularities, together with accompanying mass insertions, is equivalent to the insertion of vertex operators of the Toda field with momentum in the fundamental weight equal to these masses. In this way we relate to Toda correlation functions. They can be computed in the large N limit following the methods described above. Resolving the A_{n-1} singularities is captured by momentum flow in the intermediate channel and is obtained by bringing down appropriate number of Toda potential insertions from the action. These insertions in turn opens up cuts leading to conifold like transitions in the geometric setup.

2.6. Some simple examples

As we sketched, the relations of gauge theories, Toda systems and generalized matrix models is universal and can be applied to general curves. Here we will illustrate this construction with some simple examples of three and four vertex insertions on the sphere. Although we haven't shown yet how the background charge of the Toda field can be generated — we will return to this in section 5 — this will not change the geometric picture.

Let us first turn to the three point function. In the gauge theory this corresponds to the T_2 geometry (pairs of pant). In this case we have two insertions in the z-plane, say of mass m_1 at $z = 0$ and of mass m_2 at $z = 1$. This implies a mass at $z = \infty$ of m_0, with $m_0 + m_1 + m_2 + N = 0$. In the Liouville model we are computing the correlation function

$$\langle V_{m_0}(\infty) V_{m_1}(0) V_{m_2}(1) \rangle.$$

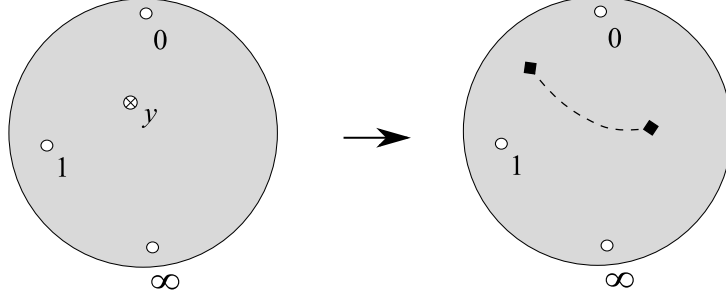in the scaling limit $m_i = \mu_i/g_s$, $g_s \to 0$.

Fig. 3: The effective geometry of the three point function in Liouville is a double cover of the sphere with a single branch cut.

The potential is a double Penner model,

$$W(z) = \mu_1 \log \Phi + \mu_2 \log(\Phi - 1).$$

There is a single critical point $z = y$ solving

$$W'(z) = \frac{\mu_1}{z} + \frac{\mu_2}{z - 1} = 0.$$

As illustrated in fig. 3, by condensation of the eigenvalues of the matrix model or the screening charge insertions, the double point $z = y$ of the curve $x^2 = W'(z)^2$ will be blown up to the form

$$x^2 = \frac{Az^2 + Bz + C}{\left(z(z-1)\right)^2}$$

Here the coefficients A, B, C are determined by the three masses μ_0, μ_1, μ_2. They also determine the size of the single cut by

$$\nu = \sum_{i=0}^{2} \mu_i.$$

So we see there are two branch points and the proper place for the chiral Liouville field $\phi(z)$ is the double cover of the sphere. Since the cover still has genus zero, this introduces no Coulomb parameters.

We can similarly consider the four point function. Here we have three logarithms

$$W(z) = \mu_1 \log \Phi + \mu_2 \log(\Phi - 1) + \mu_3 \log(\Phi - q).$$

Of course there is also a mass μ_0 at infinity. The position q gives the UV gauge coupling of the gauge theory.
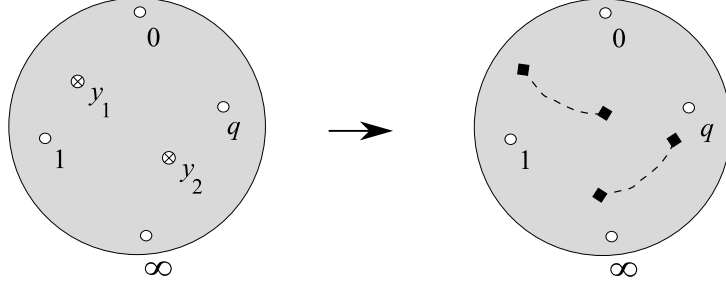
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Fig. 4: The four point function gives a spectral curve with two branch cuts. Now the cover is an elliptic curve.

There are now two critical points y_1, y_2 that in the $1/N$ expansion each open up to a branch cut of size ν_1, ν_2, as illustrated in fig. 4. Here

$$\nu_1 + \nu_2 = \sum_{j=0}^{3} \mu_j.$$

The double cover

$$x^2 = \frac{P_4(z)}{\big(z(z-1)(z-q)\big)^2}$$

now has genus one. The 5 parameters in P_4 correspond to the 4 masses μ_i together with the Coulomb parameter a, which is given by the period integral

$$a = \nu_2 - \nu_1 = \oint_{\mathcal{C}} x\, dz$$

around a contour $\mathcal{C}$ that encircles the two cuts in the form of a figure 8.

3. Local Calabi-Yau Geometries and Large N Transitions

Beginning in this section we move on to the more conceptual derivation of AGT conjecture based on B-brane probes. In this section we review the relevant local CY geometries and geometric transitions induced by branes for topological strings. We will turn to the special case of an A_{n-1} singularity in the next section.

3.1. Conifold transitions

To set the stage let us start with the simplest case of a local transition which is relevant for us: the conifold transition for a Calabi-Yau threefold. We consider both the A-model and B-model versions.
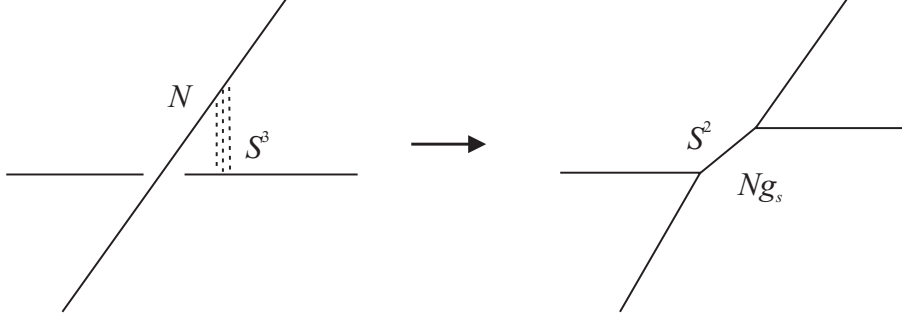
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Fig. 5: The geometric transition in the A-model that resolves the conifold: an S^3 with N branes shrinks and an S^2 of size $g_s N$ appears.

The singular conifold is given by

$$uv + xz = 0.$$

In type IIA, one considers the deformed conifold given by

$$uv + xz = \mu.$$

This local CY geometry has topology T^*S^3. One then wraps N Lagrangian A-branes over S^3. At large N this geometry is equivalent to topological strings on the resolved conifold, where the S^3 has shrunk and instead an S^2 has blown up [12], see fig. 5. The Kahler class of this S^2 is given by Ng_s, where g_s is the topological string coupling constant. The explanation of this phenomenon is that the Lagrangian N brane produce N units of 'flux' through the S^2 surrounding the S^3. This flux is measured by the Kahler form. A first principle derivation of this duality has been proposed in [26].

The open string degrees of freedom on the N Lagrangian A-branes gives rise to a $U(N)$ Chern-Simons theory on S^3 [27]. Thus the large N duality predicts that the partition function of closed topological A-model on resolved conifold is equal to the Chern-Simons partition function on S^3. This statement has been checked in great detail. Moreover, the duality also works at the level of observables [28].

In this context the observable on the open string side are Lagrangian brane probes, intersecting S^3 over links, leading to Wilson loop observables in the CS theory, which on the gravity side are given by D-brane probes of the resolved conifold geometry.
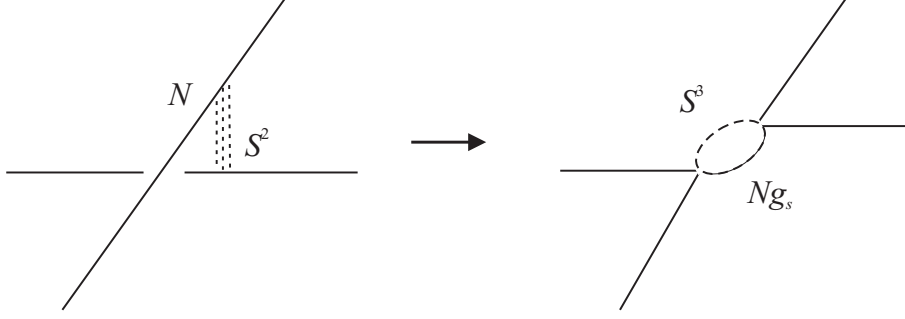
Fig. 6: In the B-model version the conifold gets deformed. Now an S^2 shrinks and an S^3 appears.
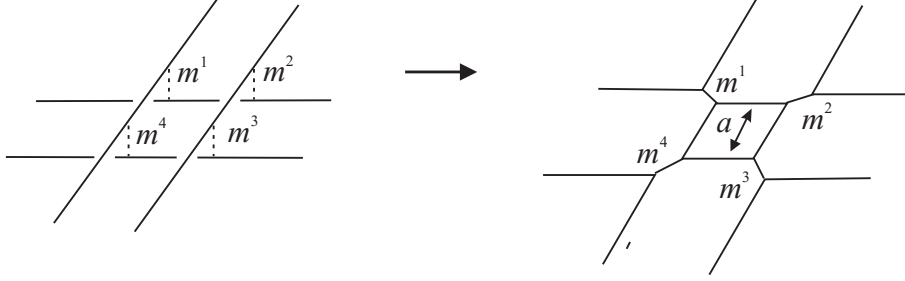

Fig. 7: Geometry that gives rise to a $SU(2)$ gauge theory with four doublets with masses $m_1, \ldots, m_4$ and Coulomb parameter a.

The same transition can also be described in the B-model setup, which is of more interest for the present paper. Namely in that context one considers N B-branes wrapped over the S^2 of the resolved conifold geometry, given by the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over the Riemann sphere. In the large N limit the S^2 is replaced by the S^3 of the deformed conifold geometry, where the integral of the holomorphic 3-form over S^3 is given by Ng_s (see fig. 6). The open string side is now given by a matrix model, as we reviewed in the last section, and the closed string side is given by the B-model Kodaira-Spencer gravity on the deformed conifold. When reduced to the underlying curve the dynamical field is the scalar $\phi(z)$.

One can also consider more interesting transitions as were studied in particular in [17]. In such cases the geometry on the A-model side was simply a collection of deformed conifold geometries without any 2-cycles. Thus one obtains a Chern-Simons gauge theory with a product group. There is an interaction between the gauge factors given by annulus diagrams, which lead to insertion of Wilson loop operators in the product CS theory.
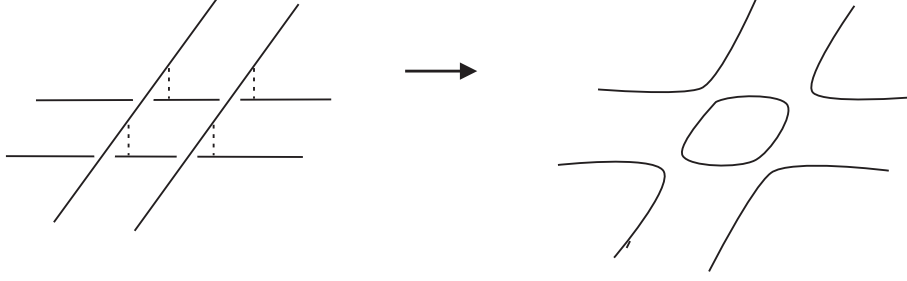
Fig. 8: The B-model version of the transition of fig. 7.

This class of theories is particularly useful for finding the large N dual for theories which in 4 dimensions lead to conformal quivers of A-type. For example, the geometry depicted in fig. 7 gives rise to an $SU(2)$ gauge theory with four doublets. To see this note that the CY geometry after transition is an A_1 singularity fibered over a $\mathbf{P}^1$ with four blow-ups. By geometric engineering, each blow-up corresponds to a hypermultiplet in the fundamental of $SU(2)$, whose mass m_i is given by the blow up parameter. One can more generally consider a product of k factors of $SU(n)$'s with bifundamental matter fields between them, except at the two ends which we obtain an additonal n hypermultiplet on each end, in the fundamental of each of the two end $SU(n)$'s. These geometries after transition correspond to 'chopping' the A_{n-1} geometries by $k+1$ lines.

Similarly we can consider the B-model version of these, which give rise to the transition depicted in fig. 8.

3.2. D-brane probes of CY geometries

D-branes as probes of CY geometries have played a key role in understanding the dynamics of topological strings. In the type IIA setup these brane probes correspond to picking a non-compact Lagrangian cycle inside the Calabi-Yau and wrap a D6 or D4 brane on it, where the extra $4/2$ dimensions of the brane fill a $4/2$-dimensional subspace of spacetime [28]. These brane probes were useful in describing Wilson loop observables for the Chern-Simons theory. A special class of such Lagrangian branes and their effective action is what gave rise to the topological vertex [16], which in turn led to a solution of topological A-model for toric 3-folds. The mirror of these geometries would correspond to the B-model where D5/D3 branes wrap a complex curve in the Calabi-Yau and fill a $4/2$-dimension subspace of spacetime. Since we will be mainly interested in the IIB setup we will specialize to that case.

As discussed in the previous section, in the type IIB setup the local geometries of interest are of the form

$$uv + F(x, z) = 0.$$

The interesting D-brane probes of this geometry correspond to choosing the following complex 1-dimensional subspace:

$$u = 0, \ v = \text{arbitrary}, \ F(x_0, z_0) = 0.$$

The internal geometry of brane probe is one complex dimensional (given by varying v) and its moduli is given by a point (x_0, z_0) on the $F(x, z) = 0$ curve. Consider the local coordinate on the curve and denote it by z (where x is solved in terms of z using $F(x, z) = 0$).

It has been shown that these branes are described by a *free* fermionic system [15]. The fermion field $\psi(z)$ descirbes the creation of a D-brane stretching in the v-direction. Similarly the conjugate field $\psi^*(z)$ describes the creation of a D-brane in the u-direction. Moreover, the Riemann surface is 'quantum' in the sense that on the x-z plane there is a non-commutativity [15,29,30] given by

$$[x, z] = g_s,$$

where g_s is the topological string coupling constant. This implies that for the branes, whose position is given locally by z, the variable x is not quite a classical value, but it is an operator given by $g_s \cdot \partial/\partial z$. Note that this implies that x can be identified with the Hamiltonian in the *chiral* path-integral where we view z as the chiral 'time'. In other words the chiral path-integral would be given by

$$\frac{1}{\hbar} \int H \, dt \to \frac{1}{g_s} \int x \, dz \tag{3.1}$$

Viewing the operation by x as the generator of z translations, has implications for correlation functions of the branes. In particular using this, one concludes that the equation for the curve becomes an operator acting on the partition function of the brane:

$$F(x = g_s \frac{\partial}{\partial z}, z) \cdot \left\langle \cdots \psi(z) \cdots \right\rangle = 0,$$

where this is defined up to normal ordering ambiguities in F. Moreover, D-brane probes shift the value of the holomorphic 3-form Ω by $N g_s$, where N is the number of D-branes.

In other words, Ω measures the 'flux' of the internal 1-complex dimensional branes (which can be surrounded by a 3-cycle). If we bosonize the fermions as $\psi = e^{i\phi}$, then Ω is represented by the one-form $\partial\phi$ on the curve.

Since x acts on the brane probe $\psi(z)$ as

$$x \cdot \psi(z) = g_s \frac{\partial\psi}{\partial z},$$

bosonization implies that

$$x \cdot \psi(z) = g_s(\partial\phi)\psi(z).$$

This in particular implies that if we consider a sector where ϕ has a momentum p/g_s, then ψ is an eigenstate of x with eigenvalue p.

Note that if a brane probe goes through a 1-cycle around which we are measuring the period of $\partial\phi$, it can change the period, if its path intersects it. Consider in particular the deformed conifold geometry, represented as the curve

$$F(x, z) = xz - m = 0$$

By a change of variables we can view this as

$$x^2 = z^2 - m$$

This means that we can view the x plane as a double cover of the z plane, with a cut running between $z = \pm\sqrt{m}$. The conifold size is measured by

$$\oint_{cut} x dz = m$$

(up to normalization factors which we ignore here). Now consider moving k B-branes from infinity and through the cut to the other sheet and taking them to infinity of the other sheet. Then after this operation we have

$$\oint_{cut} x dz = m + k g_s.$$

4. A_{n-1} Geometries and D-brane Probes

Following these general remarks we will now turn to the geometries that lead to conformal gauge theories of the type considered in section 3.

4.1. The non-commutative geometry of A_{n-1} singularities

We are interested in CY backgrounds described by a family of A_{n-1} singularities fibered over a Riemann surface Σ. These geometries are T-dual to a 5-brane wrapping Σ and extending in space-time. They corrspond to supersymmetric gauge theories in four dimensions with $SU(n)$ gauge group. Locally, the geometry will be given by

$$uv = x^n,$$

where this singularity has a locus given by Σ, which we parametrize by a coordinate z. Note that the holomorphic 3-form is given by

$$\Omega = \frac{du}{u} \wedge dx \wedge dz.$$

In particular dx transforms as a section of K_Σ^{-1}, the inverse of the canonical bundle on the surface, so that the two-form $dx \wedge dz$ has a section over Σ.

To describe a brane probe in this geometry we choose a point on the space parametrized by (x, z) that satisfies $x^n = 0$. In other words, we choose any point on the surface Σ and in addition choose a module for the variable x, such that it satisfies the relation $x^n = 0$. Such a module is naturally labelled by a state ψ_1 which is the 'lowest' state with x-charge. In other words, ψ_1 generates the rest of the module by the action of x on it (the module is cyclic)

$$\psi_1, x\psi_1, ..., x^{n-1}\psi_1.$$

So at each point $z \in \Sigma$ we have an n-dimensional module given by the basis

$$\psi_i(z) = x^{i-1}\psi_1(z), \qquad i = 1, \ldots, n.$$

These fermions $\psi_i(z)$ represent the n choices of branes. In analogy with minimal LG theories these branes can be identified with the vacua of RR sector and x is the chiral field acting on them. In this sense we can think of ψ_1 as the state with the lowest R-charge. At a point z the action of x is given by the $n \times n$ matrix

$$x = \begin{pmatrix} 0 & 0 & \cdots & & 0 \\ 1 & 0 & \ddots & & \vdots \\ 0 & \ddots & \ddots & & \\ \vdots & \ddots & & & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Note that in terms of its action on the fermion states at position z, the operator x is represented by the current

$$J_+(z) = \sum_{i=1}^{n-1} J_+^i = \sum_{i=1}^{n-1} \psi_i^* \psi_{i+1},$$

because

$$\oint_z dw \, J_+(w) \cdot \psi_i(z) = \psi_{i+1}(z).$$

However, this cannot be the full story. Taking quantum corrections of the topological string into account, we know that x gets an extra contribution $g_s \partial/\partial z$. This seems to be in contradiction with the classical statement that x maps one brane to another as

$$x : \quad \psi_i(z) \rightarrow \psi_{i+1}(z).$$

Combining these two ingredients, we see that in the quantum theory x should be represented by a covariant derivative ('Drinfeld-Sokolov connection')

$$x = g_s D_z, \qquad D_z = \partial_z + \frac{1}{g_s} J_+(z).$$

Given the relation (3.1) this implies that the chiral path-integral, due to this covariantization, picks up an additional term given by the insertion of

$$\exp \int dz \, J_+(z).$$

This can be seen as a background gauge field that restores the action of x on the path integral. In other words, now there is an 'internal' action of the field x on the branes captured by the non-trivial connection J_+. It is the presence of J_+, and mainly this, that reflects that we are dealing with an A_{n-1} singularity. Note that the screening operator is naturally mapped to the chiral part of the A_{n-1} Toda potential:

$$J_+ = \sum_{i=1}^{n-1} \psi_i^* \psi_{i+1} = \sum_{i=1}^{n-1} e^{i(\phi_{i+1} - \phi_i)}.$$

As is usual with chiral Hamiltonians the meaning of the chiral action is a bit imprecise. To make it more precise we note that the usual way is to consider insertions of $\int J_+$ on cyclinders and by gluing rules extend it to higher genera.
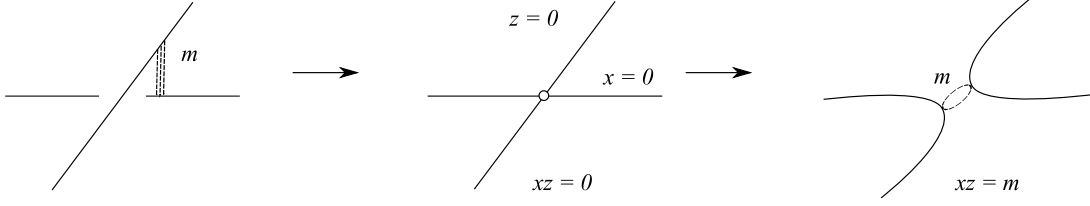
Fig. 9: Putting m B-branes on the resolved conifold deforms the curve $xz = 0$ to $xz = m$.

It is convenient to also describe the deformations of the A_{n-1} singularity in this formalism. If we bosonize the fermion $\psi_i = e^{i\phi_i}$ as before, the above argument shows that the momentum $\partial\phi_i$ corresponds to the eigenvalue of the action of the operator x on the corresponding brane. Here it is important to consider the brane in the background of the screening operators $\exp \int J_+$. In this case the action of x on the ψ_i is again strictly given by the $\partial\phi_i$ insertion. Thus x acting on the ith brane has eigenvalue p_i. Therefore we see that x acting on the collection of branes satisfies

$$\prod_{i=1}^{n}(x - p_i) = 0,$$

which describes the general deformation of the A_{n-1} singularity. We thus see that the Coulomb parameters of the $U(n)$ gauge theory get naturally identified with the momenta of the corresponding scalar fields. In particular if we write the above equation as

$$\sum_{k=0}^{n} x^{n-k}\sigma_k(p) = 0,$$

then

$$\sigma_k(p) = \sum_{distinct\ i_r} p_{i_1}\cdots p_{i_k} = \sum_{distinct\ i_r} \partial\phi_{i_1}\cdots\partial\phi_{i_k} \tag{4.1}$$

4.2. Chopping the A_{n-1} Geometries and D-brane probes

Let us now turn our attention to the inclusion of D-branes. Let us consider the A_0 case first. This is given by the local geometry of the conifold

$$uv + xz = 0.$$

The corresponding curve is given by $F(x, z) = xz = 0$. This is depicted in fig. 9.

Before the transition we put m B-branes wrapping the $\mathbf{P}^1$ that appears in the resolution of the conifold singularity. These branes create after the transition a hypermultiplet of mass m. After the transition the curve is changed into

$$xz = m.$$

We are interested in the probe brane living on the z-plane. From the perspective of the probe we have m units of flux at $z = 0$ created by the B-branes. As mentioned before, the flux is measured by the holomorphic 3-form, which in the local curve model is captured by the field $\partial\phi(z)$ with expectation value xdz. Thus we see that the effect of m B-branes is to lead to a flux

$$\oint_{z=0} \partial\phi = mg_s.$$

In other words, this corresponds to the insertion of the operator $e^{im\phi}$ at $z = 0$. More generally, if we considered the geometry given by $F(x, z) = x(z - q) = 0$ and placed m B-branes ending at $z = q$ we would have an insertion of $e^{im\phi(q)}$.

Now we consider the A_{n-1} geometries, "chopped off" at some values of z, in other words local geometries of the form

$$F(x, z) = x^n(z - q) = 0.$$

One can ask what is the effect of the mass insertion on the transition. Locally this can also be described in the following way. Let us first concentrate on the $n = 2$ case. If $m = 0$, the geometry is simply given by

$$x^2(z - q) = 0.$$

However, the mass deformation will induce a transition in the geometry near $z = q$. It is not difficult to find out wat this effect should be, namely it should be the mirror of the local blow up. We claim it is given by
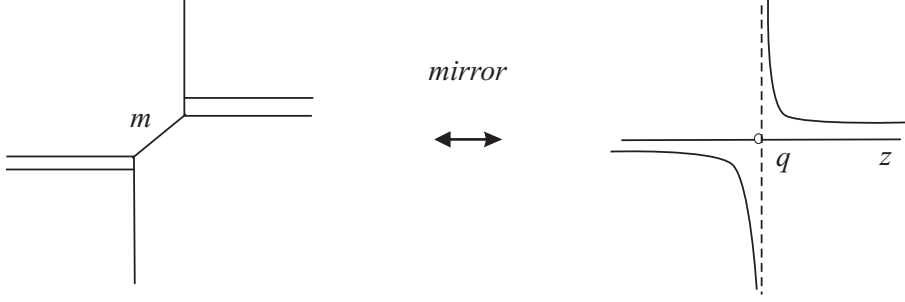
$$x^2(z - q) - mx = 0.$$

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Fig. 10: Putting m branes on the A_1 curve that describes a $SU(2)$ gauge theory, gives the curve $x^2(z-q) - mx = 0$.
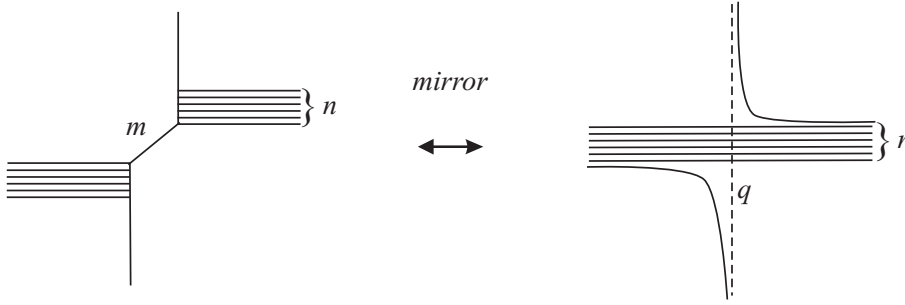

Fig. 11: The generalization of fig. 10 to a A_{n-1} curve that describes a $SU(n)$ gauge theory. Now the curve is given by $x^n(z-q) - mx^{n-1} = 0$.

To see this, note that if we consider the curve $x^2(z-q) - mx = 0$, and study how x varies as a function of z, we see that for large values of z the two roots are on top of each other. However, near $z = q$ the roots are split. The mirror geometry is fixed uniquely by having the correct asymptotics of the spectral curve (matching the A-model toric geometry). This is the geometry depicted in fig. 10.

Using the relation of momenta to the expectation values of $\partial\phi_i$ given in (4.1) we see that this means $\partial\phi_1 + \partial\phi_2 = m/(z-q)$, but $\partial\phi_1 \cdot \partial\phi_2 = 0$. This mass can be induced by inserted the operator $\exp(im\phi_1(q))$. (The other choice can be obtained from this by permutation action.)

Similarly we can consider the more general A_{n-1} case. In that case the B-model mirror for the mass insertion is uniquely fixed by matching the A-model asymptotics (see fig. 11), and is given by

$$x^n - \frac{m}{z-q}x^{n-1} = 0$$

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which is again induced by the insertion of $\exp(m\phi_1(q))$ in the the path integral. Note that this agrees with the prescription suggested in [10].

It is sometimes convenient to shift variables, to write the curves in a slightly different form. For example, note that for the A_1 case we can shift $x \to x - \frac{1}{2}\frac{m}{z-q}$. Then the curve can be written as

$$x^2 = \left(\frac{m}{2(z-q)}\right)^2.$$

In a similar vein we see that in the presence of several masses m_i at positions q_i we effectively have a different CY geometry after transition given by

$$uv + x^2 - x\sum_i \frac{m_i}{z - q_i} = 0.$$

In the case of genus zero this is exact, *i.e.*, here the Green's functions are simple and given by $1/(z - q_i)$.[4] By a change of variables $x \to x - \sum_i \frac{1}{2}\frac{m_i}{z-q_i}$ we end up with the geometry

$$uv + x^2 - W'(z)^2 = 0,$$

where

$$W(z) = \sum_i \frac{m_i}{2}\log(z - q_i).$$

We thus end up with a local Calabi-Yau which has conifold like singularities at the points where $W'(z) = 0$.

However, this geometry is still not the most general geometry relevant for the gauge theory in question, because we would be interested in turning on arbitrary Coulomb branch parameters. That will resolve the conifold singularities. These turn out to correspond to bringing down the screening operators $\int J_+$. This naturally connects to the matrix model description discussed in section 2 where we found that the insertion of $\int J_+$ is related to placing branes (matrix eigenvalues) at the critical points of W.

We have thus receovered the equivalence of the Brane probe theory with the Toda theory together with the correct dictionary between them. Namely we have a $U(n)$ Toda theory with mass insertions given by fundamental weight of Toda. This derivation was done for the case where the Nekrasov deformation is turned off. In the next section we extend this derivation to this deformation which leads to turning on the background charge for Toda.

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[4]   It is easy to generalize these statements for higher genus.

5. Nekrasov's Deformation and β-Ensembles

We will now generalize the above relations between topological strings, matrix models and Toda theories to include Nekrasov's deformation of the topological strings. From the point of the four dimensional gauge theories this corresponds to working equivariantly with respect to the $U(1) \times U(1)$ rotation group inside the $SU(2) \times SU(2) \cong SO(4)$ acting on the space-time $\mathbf{R}^4$. The corresponding equivariant parameters ϵ_1, ϵ_2 have a physical interpretation as the angular velocities of these rotations. The geometric implementation involves the so-called Ω background [5]. It is known that this corresponds to turning on $SU(2)_R$ holonomies on the internal Riemann surface Σ in order to preserve the topological supersymmetry, that otherwise would be broken by turning on the ϵ_i's. More precisely: Nekrasov's deformation corresponds to a two-form in $\mathbf{R}^4$ with self-dual component proportional to $\epsilon_1 - \epsilon_2$ and anti-self-dual component proportional to $\epsilon_1 + \epsilon_2$. So, if $\epsilon_1 \neq -\epsilon_2$ supersymmetry is broken.

Before we turn to the implications for topological string theory and matrix models, let us first introduce some convenient notation to parametrize the Nekrasov deformation. We will use the following notation:

$$b^2 = \frac{\epsilon_1}{\epsilon_2} = -\beta,$$

and

$$g_s^2 = -\epsilon_1 \epsilon_2.$$

The Liouville background charge is then expressed as

$$Q = b + \frac{1}{b} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} + \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{\epsilon_1 + \epsilon_2}{g_s}.$$

Note that the duality symmetry $\epsilon_1 \leftrightarrow \epsilon_2$ corresponds to $\beta \leftrightarrow 1/\beta$ and $b \leftrightarrow 1/b$. The undeformed case is obtained at $\epsilon_1 = -\epsilon_2 = g_s$, $b = i$, $Q = 0$.

5.1. Ω backgrounds and Toda field background charges

Let us first consider the implications for this deformation for the geometry of topological strings. We interpret Nekrasov's Ω background in the present context as follows: Let us consider the spacetime to be $\mathbf{R}^4 = \mathbf{C}^2$ with coordinates z_1, z_2. Consider the canonical

line bundle of this complex spacetime represented by the section $dz_1 \wedge dz_2$. The local geometry of Calabi-Yau over the Σ is

$$\mathcal{N}_\Sigma \oplus \mathbf{C}^2.$$

In other words, we view $\mathbf{C}^2$ also as part of the 'internal' geometry. We are interested according to [5] in considering a non-trivial bundle structure of $\mathbf{C}^2$ over Σ. Let $F_{\mathbf{C}^2}$ represent the curvature of the canonical line bundle of $\mathbf{C}^2$ over Σ. We take this to be represented by a multiple of the curvature on Σ:

$$F_{\mathbf{C}^2} = -\epsilon \cdot R_\Sigma,$$

where we will identify $\epsilon = \epsilon_1 + \epsilon_2$. Now, however, the Calabi-Yau condition is lost and we have lost supersymmetry. To restore it, we need the $\mathcal{N}_\Sigma$ to also twist. The trace of the rotation generator in $\mathcal{N}_\Sigma$ we will denote by $U(1)_R \subset SU(2)_R$. We need the curvature of this bundle to cancel this additional curvature. In other words we need to change this curvature so that

$$\delta F_{U(1)_R} = \epsilon \cdot R_\Sigma.$$

Let J_R denote the $U(1)_R$ current acting on fields living on Σ. This means that we need to add to the action of these fields an additional term given by

$$(\epsilon_1 + \epsilon_2) \int_\Sigma J_R \wedge \omega_\Sigma$$

where ω_Σ represents the spin connection on Σ.

To implement this deformation we need to identify J_R in our context. Note that the $U(1)_R$ can be identified with the phase of the holomorphic 2-form $dx \wedge du/u$ on the geometry normal to Σ. This corresponds to action

$$x \;\to\; \exp(i\theta) \cdot x,$$

where the phase θ is given by

$$\theta = \frac{\epsilon_1 + \epsilon_2}{g_s} = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}}.$$

Such an action corresponds to the following transformation on the fermions

$$\psi_k \;\to\; \exp\left(i(k - 1 + const)\theta\right) \cdot \psi_k,$$

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using the fact that the ψ_k are obtained from ψ_1 by the action of x^{k-1}. The constant here depends on the choice of the action of $U(1)_R$ on ψ_1. A natural choice for this constant, suggested by CTP invariance of the analog LG models, is $-(n-1)/2$. In other words, this is like twisting the LG model and therefore leads to the background charge coupling given by

$$S = \ldots + i\theta \int \sum_k Q_k \partial \phi_k \wedge \omega_\Sigma = \ldots + i\theta \int \sum_k Q_k \phi_k R^{(2)} \sqrt{g},$$

where $Q_k = (k - (n+1)/2)$ and

This implies that we have a term proportional to

$$\int (b + 1/b) \sum_k (k - (n+1)/2) \phi_k R^{(2)} \sqrt{g},$$

where $b = \sqrt{\epsilon_1/\epsilon_2}$. We thus see we have obtained the Toda system with background charge, simply by considering twisting of the normal geometry over Σ.

5.2. Generalized matrix models and β ensembles

We will now turn to the corresponding deformation in the matrix model, which will turn out to be given by the so-called β-ensemble. Again, as in section 2, for simplicity we restrict to the $SU(2)$ case, but generalizations to $SU(n)$ are straightforward.

As we have just seen, including general ϵ_1, ϵ_2 corresponds to a background charge Q for the scalar field ϕ. This changes the conformal dimensions of the vertex operators. In particular, the screening charge (of dimension 1) is now given by

$$J_+ = e^{b\phi}.$$

Correlations of the Coulomb gas are now given by

$$\langle J_+(z_1) \cdots J_+(z_N) \rangle = \prod_{I<J} (z_I - z_J)^{-2b^2} = \Delta(z)^{-2b^2}.$$

Replacing the usual second power of the Vandermonde determinant by this measure is known in the matrix model literature as the β-ensemble [19,20]. These generalized matrix models are given by eigenvalue integrals of the form

$$Z = \int d^N z \, \Delta(z)^{2\beta} \cdot \exp \sum_I \frac{\beta}{g_s} W(z_I).$$

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Clearly, we can map the Liouville model directly to the β-ensemble if we identify

$$\beta = -b^2.$$

These generalized measures are traditionally studied because they naturally appear if, instead of a unitary ensemble of hermitean matrices, one considers orthogonal or symplectic matrices. More precisely, the β-ensemble reproduces these alternative marix models for

$$SO(N): \ \beta = \frac{1}{2}, \qquad Sp(N): \ \beta = 2.$$

Note that for these values the duality symmetry $\beta \to 1/\beta$ interchanges

$$SO(N) \leftrightarrow Sp(N),$$

which is not an unfamiliar symmetry for string theory.

We claim that for any geometry on which the topological string can be described by a matrix model, generalizing the measure to the β-ensemble immediately gives the refined invariant as a function of ϵ_1, ϵ_2. This becomes already clear by considering the case of a Gaussian measure which corresponds to the conifold geometry. We find

$$Z = \int d^N z \ \Delta(z)^{2\beta} \cdot \prod_I e^{-z_I^2}$$

This integral is a special case of the Selberg integral and can be evaluated to be

$$Z = (2\pi)^{N/2} \prod_{k=1}^{N} \frac{\Gamma(1+\beta k)}{\Gamma(1+\beta)}.$$

This gives for the free energy in the large N limit

$$F = \log Z = \int \frac{ds}{s} \frac{e^{\mu s}}{(1 - e^{\epsilon_1 s})(1 - e^{\epsilon_2 s})}$$

with $\mu = g_s N$. This is indeed the contribution of a single hypermultiplet of mass μ. Note that we recognize in this expression the partition function of the $c = 1$ string at radius b. This therefore generalizes the usual identification of the conifold with the $c = 1$ string at self-dual radius for $\beta = -b^2 = 1$. The symmetry $b \to 1/b$ is also obvious.

To a large extent the methods that are used to solve traditional matrix models can be generalized to the β-ensemble. For example, the loop equations again are a reflection of diffeomorphisms $\delta z = z^{n+1}$ acting on the eigenvalues as generated by the modes L_n of

the stress-tensor $T(z)$. However, because of the 'funny' β-measure, the stress-tensor is no longer quadratic in terms of the collective field

$$\partial\phi = W'(z) + \sum_I \frac{g_s}{z_I - z}.$$

It picks up a background charge [20]

$$T(z) = \tfrac{1}{2}(\partial\phi)^2 + Q\partial^2\phi,$$

with $Q = b + 1/b$. This is of course perfectly consistent with the connection to Liouville theory that we have argued for. It can be seen as a confirmation that we are implementing the Nekrasov deformation correctly in the topological string framework.

So, combining all ingredients we see that the β-ensemble can be mapped to a chiral version of Liouville with general central charge $c = 1 + 6Q^2$

$$\int d^2z \left(\partial\phi\overline{\partial}\phi + Q\phi R^{(2)}\sqrt{g} \right) + \int dz\, e^{b\phi}.$$

The presence of a background charge does not influence the leading solution of the matrix model, which is captured by the quadratic part of $T(z)$. The genus zero contribution $\mathcal{F}_0$ is therefore still described in terms of the spectral curve (2.2). However, this is not rue for contributions with a different topology. Because $T(z)$ is no longer invariant under $\phi \to -\phi$, the partition function as a function of g_s now also receives odd contributions

$$Z = \exp F, \qquad F = \sum_{n \geq 0} g_s^{n-2}\mathcal{F}_{\frac{n}{2}}(\beta).$$

This is well-known for the SO/Sp matrix models at $\beta = \tfrac{1}{2}, 2$ where unoriented surface contribute. For example $\mathcal{F}_{\frac{1}{2}}$ represents the contribution of $\mathbf{RP}^1$.

This connection can be extended to correlation functions of vertex operators. They take the form

$$V_m(q) = e^{m\phi/2} = \prod_I (z_I - q)^{-bm}$$

Thus we have to analyse the expression

$$\left\langle \prod_i V_m(q_i) \right\rangle_N = \int d^N z \prod_{I<J} (z_I - z_J)^{-2b^2} \prod_{i,I} (z_I - q_i)^{-bm_i}$$

in the limit $N, m_i \to \infty$. The analysis of this model follows exactly along the same lines as in section 2 for $\beta = 1$.

6. Directions for Future Research

In this paper we have shown that the AGT conjecture about the chiral blocks of Toda CFT being related to $\mathcal{N} = 2$ gauge theories, arises naturally from its interpretation in the context of topological strings. A key ingredient in this derivation is the large N limit of branes in topological strings and the geometric transitions they induce. We gave two (not completely unrelated) derivations: One was more conceptual, based mainly on the general properties of B-branes and their relation to free fermionic systems and non-commutative geometry. The other was more specific to the relation between topological strings and matrix models. In particular we related this to the matrix models of 'multi-Penner' potentials for the case of sphere.

Let us now mention some points for further study. First of all, it is important that the SW geometry arises for us holographically in the large N limit of the brane probe theory, even for A_1. There is yet another sense of large N limit one can consider in this context, which is the A_{n-1} geometry in the limit of large n. The gravity version of this has been studied in [8]. Combining this with our approach we have a double N holography. In the context of matrix models, this corresponds to the large N limit of A_∞ quiver matrix models. As discussed in [31] this corresponds to matrix quantum mechanics. It would be interesting to develop this limit of the theory from this viewpoint.

Second, it is clear that the D and E series can be studied along the lines suggested here. In particular the corresponding matrix models would be a quiver of the D or E type. It should be interesting to develop techniques to better understand these matrix models, or equivalently the corresponding Toda correlations.

Third, we have interpreted Nekrasov's Ω background in a specific geometric way in topological strings based on a local curve. It would be intersting to study this further, and in particular consider its application to more general local Calabi-Yau geometries.

Finally, in this paper we have focused on connections between topological strings and 4d theories. However, it is well known that topological strings also captures vacuum geometry of 5d gauge theories compactified on a circle, more clearly realized in the context of type IIA on CY 3-folds in terms of toric geometry. In a sense here we have been studying only the special limit $R \to 0$ of such a system. Clearly it is important to study the R dependence of this bigger system. The most naive generalization of our results would

suggest, in the matrix model setup, to replace the Vandermonde by a q-deformed version. It would be interesting to study these issues further.

Acknowledgements

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